1. Simple Harmonic Motion

1.1 Free oscillation

Suppose we choose as a model oscillator the mass, m, suspended on a spring of stiffness k. Then we are to solve Newton's second law, force = mass × acceleration, as a differential equation,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

which we write

$$\ddot{x} = -\omega_0^2 x \qquad (1)$$

by using two dots to indicate a *second derivative with respect to time.* We will use one dot to indicate the first derivative. We have also combined the two constants, m, the mass and k, the spring constant, to define an *angular frequency*,

$$\omega_0^2 = \frac{k}{m}$$

We're not mathematicians, we just want a solution of this thing; so try $x = Ae^{st}$. Then by simple differentiating, we have

$$x = Ae^{st}\ ;\quad \dot{x} = sAe^{st}\ ;\quad \ddot{x} = s^2Ae^{st}$$

We only have to put this back into (1) to see that

$$s^2Ae^{st} + \omega_0^2Ae^{st} = 0 \longrightarrow s^2 + \omega_0^2 = 0 \longrightarrow s = \pm\mathrm{i}\omega_0$$

So we have two solutions:

$$x = Ae^{\mathrm{i}\omega_0\mathrm{t}} \text{ and } x = Ae^{-\mathrm{i}\omega_0\mathrm{t}}$$

The theory of second order, linear differential equations tells us that the most general solution is a linear combination of the two solutions with two arbitrary coefficients, that we will call A_1 and A_2:

$$\begin{aligned}
x &= A_1e^{\mathrm{i}\omega_0\mathrm{t}} + A_2e^{-\mathrm{i}\omega_0\mathrm{t}} \\
&= (A_1 + A_2)\cos\omega_0 t + \mathrm{i}(\mathrm{A}_1 - \mathrm{A}_2)\sin\omega_0\mathrm{t} \\
&= A\cos\omega_0 t + B\sin\omega_0 t \qquad \text{(a)} \\
&= C\cos\phi\sin\omega_0 t + C\sin\phi\cos\omega_0 t \qquad \text{(b)} \\
&= C\sin(\omega_0 t + \phi)
\end{aligned}$$

In going from line (a) to line (b) I have changed from the variables A and B to variables C and ϕ by making these two definitions,

$$A = C\sin\phi \text{ and } B = C\cos\phi$$

because then I can use the usual formula for $\sin(a + b)$ to arrive at the last line.

Now what we have is

$$x = C\sin(\omega_0 t + \phi)$$
$$\dot{x} = v = C\omega_0\cos(\omega_0 t + \phi)$$

To fix the, up to now arbitrary, constants requires us to know "boundary conditions." Let's suppose that at $t = 0$, $x = x_0$, say, and $v = v_0$, the initial velocity. These conditions give,

$$x_0 = C\sin\phi\ ,\quad \sin\phi = \frac{x_0}{C} \tag{c}$$
$$v_0 = C\omega_0\cos\phi\ ,\quad \cos\phi = \frac{v_0}{C\omega_0} \tag{d}$$

Now, square and add (c) and (d),

$$C = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$$

and divide (c) by (d)

$$\phi = \arctan\frac{x_0\omega_0}{v_0}$$

Finally, if we start off the oscillator at $t = 0$ with $v_0 = 0$ and $x = x_m$, for example we pull out the spring to maximum deflection, x_m, hold it still ($v_0 = 0$) and let it go; then the solution is

$$x = x_m\sin(\omega_0 t + \frac{1}{2}\pi) = x_m\cos(\omega_0 t)$$

1.2 Damping

To the differential equation (1), which is after all Newton's second law—force equals mass times acceleration—we add an additional force, $-b\dot{x}$; b is called the "damping coefficient". This force is *proportional to the velocity*, which is what you'd expect. Try swimming in syrup: the faster you swim the bigger is the drag, or viscous, force. So now we need to solve the differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

which we re-write as

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0$$

We define a new constant, Z, such that

$$\frac{b}{m} = 2Z\omega_0$$

is the *frictional force per unit mass and unit speed*. Now our differential equation is

$$\ddot{x} + 2Z\omega_0\dot{x} + \omega_0^2 x = 0$$

As before we try

$$x = Ae^{st}\ ;\quad \dot{x} = sAe^{st}\ ;\quad \ddot{x} = s^2Ae^{st}$$

and so

$$s^2 + 2Z\omega_0 s + \omega_0^2 = 0$$

leads to

$$s = \omega_0 \left(-Z \pm \sqrt{Z^2 - 1} \right) \tag{2}$$

and the general solution must be

$$x = A_1 e^{st} + A_2 e^{-st} \tag{3}$$

Critical damping is defined as the condition $Z = 1$. For that case we define

$$b_{\text{crit}} = 2m\omega_0 = 2\sqrt{mk}$$

and we give a name to Z by

$$\frac{b}{b_{\text{crit}}} = Z$$

being called the *damping factor*, or *damping ratio.*

Underdamping is the condition $Z < 1$ or $b < b_{\text{crit}}$. This is usually the most interesting case, and for which

$$Z^2 - 1 < 0$$

meaning that there are two roots to (2), namely,

$$s_1 = \omega_0 \left(-Z + \mathrm{i}\sqrt{1 - \mathrm{Z}^2} \right)$$
$$s_1 = \omega_0 \left(-Z - \mathrm{i}\sqrt{1 - \mathrm{Z}^2} \right)$$

and the solution to (3) is

$$x = e^{-Z\omega_0 t} \left(A_1 e^{\mathrm{i}\sqrt{1-\mathrm{Z}^2}\omega_0 \mathrm{t}} + A_2 e^{-\mathrm{i}\sqrt{1-\mathrm{Z}^2}\omega_0 \mathrm{t}} \right)$$

We then simplify this in the same manner as for equations (a) and (b):

$$\begin{aligned} x &= Ce^{-Z\omega_0 t} \sin\left(\sqrt{1 - Z^2}\omega_0 t + \phi \right) \\ &= Ce^{-\alpha t} \sin(\omega_D t + \phi) \end{aligned}$$

where

$$\alpha = \frac{1}{2}\frac{b}{m} = Z\omega_0$$

is called the *damping constant*, and

$$\omega_D = \omega_0\sqrt{1 - Z^2} = \omega_0\sqrt{1 - \frac{1}{4}\frac{b^2}{mk}} < \omega_0$$

is the *damped frequency.*

Again, if at $t = 0$, $x = x_m$ and $v = 0$, the solution associated with these boundary conditions is

$$\begin{aligned} x &= x_m\, e^{-\alpha t}\, \sin\left(\omega_D t + \frac{1}{2}\pi\right) \\ &= x_m\, e^{-\alpha t}\, \cos\omega_D t \end{aligned}$$

1.3 Driven oscillators

In real life we are less interested in an oscillator that is oscillating at its natural freqency, ω_0, or its natural damped frequency, ω_D, than in the behaviour of an undamped or damped oscillator when we choose to drive it at some frequency, ω, that *we* choose. Situations of this phenomenon are ubiquitous in physics and engineering. Try and write down some half a dozen examples of your own.

3.1 Undamped driven oscillator

The oscillator is driven by a periodic force of angular frequency ω and amplitude F_0. That means we have one more force to add in to Newton's second law, namely

$$F = F_0 \sin\omega t$$

and force = mass × acceleration now reads

$$m\ddot{x} = F_0 \sin\omega t - kx \tag{4}$$

Eventually the oscillator has no choice but to vibrate at the frequency of the driving force, whether it likes it or not, so we must have,

$$\begin{aligned} x &= A\sin\omega t \\ \dot{x} &= A\omega\cos\omega t \\ \ddot{x} &= -A\omega^2\sin\omega t \end{aligned}$$

Equation (4) now reads

$$-mA\omega^2\sin\omega t + kA\sin\omega t = F_0\sin\omega t$$

That is,

$$\begin{aligned} A &= \frac{F_0}{k - m\omega^2} = \frac{F_0/k}{1 - \frac{\omega^2}{\omega_0^2}} \\ &= \frac{A_s}{1 - \frac{\omega^2}{\omega_0^2}} \end{aligned}$$

using

$$\omega_0 = \sqrt{\frac{k}{m}}$$

the natural frequency of the undamped oscillator. We call A_s the *static amplitude* and we call A the *dynamic amplitude*; their ratio is called the *magnification factor*,

$$D_s = \frac{A}{A_s} = \left(1 - \frac{\omega^2}{\omega_0^2}\right)^{-1}$$

If the driving frequency is less that the natural frequency the magnification factor is positive and the displacement is in phase with the driving force. Conversely if $\omega > \omega_0$, $D_s < 0$. An amplitude cannot be negative, so we'll have instead, for this case, to use the solution

$$x = -A \sin \omega t$$

which implies a phase difference of π (180°) between the displacement and the driving force. Thirdly, if $\omega = \omega_0$, $D_s \to \infty$ and we have *resonance.* In real life this never happens as there is always damping. But interesting things *do* happen when we drive an oscillator at a frequency close to its natural one.

3.1 Damped driven oscillator

Now we include the velocity dependent damping force into equation (4):

$$m\ddot{x} = F_0 \sin \omega t - b\dot{x} - kx$$

or

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin \omega t \tag{4a}$$

Eventually after transients have died away, the oscillator must vibrate at the frequency of the driving force. It may not like it and it will protest unless the driving frequency is close to the natural frequency of the undriven oscillator. Its reluctance to cooperate is reflected in a reduction in amplitude. Nearer to *resonance* the amplitude is large. The so called *resonance curve* or relation between amplitude and driving frequency is what we will be seeking in the mathematical development that follows. The oscillator will necessarily vibrate at the frequency of the driving force, but it will not necessarily be in phase with it. Hence the solution for the amplitude must look like

$$\begin{aligned} x &= A \sin(\omega t - \phi) \\ \dot{x} &= A\omega \cos(\omega t - \phi) \\ \ddot{x} &= -A\omega^2 \sin(\omega t - \phi) \end{aligned}$$

when I plug these into (4a) I get

$$\begin{aligned} m\left[-A\omega^2 \sin(\omega t - \phi)\right] + b\left[A\omega \cos(\omega t - \phi)\right] + kA \sin(\omega t - \phi) &= F_0 \sin \omega t \\ &= F_0 \sin(\omega t - \phi + \phi) \end{aligned}$$

Rearranging this I have

$$\begin{aligned} A\left(k - m\omega^2\right) \sin(\omega t - \phi) + Ab\omega \cos(\omega t - \phi) \\ = F_0 \left[\sin(\omega t - \phi) \cos \phi + \cos(\omega t - \phi) \sin \phi\right] \end{aligned}$$

Now, equate the coefficients of $\sin(\omega t - \phi)$ and $\cos(\omega t - \phi)$ and obtain

$$\begin{aligned} Ab\omega &= F_0 \sin\phi \\ A\left(k - m\omega^2\right) &= F_0 \cos\phi \end{aligned}$$

We square and add these two, recalling that $\sin^2\phi + \cos^2\phi = 1$,

$$F_0^2 = A^2\left[\left(k - m\omega^2\right) + b^2\omega^2\right]$$

which means that we have, for the *dynamic amplitude*,

$$\begin{aligned} A &= \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}} \\ &= \frac{F_0/k}{\sqrt{\left(1 - \frac{m\omega^2}{k}\right)^2 + \frac{b^2\omega^2}{k^2}}} \end{aligned}$$

We also divide our two equations to find the *phase difference*, or phase angle, ϕ, between the oscillator and its driving force,

$$\tan\phi = \frac{b\omega}{k - m\omega^2}$$

We can simplify the formulas for A and ϕ using these definitions that we have encountered already in these notes,

$$\omega_0 = \sqrt{\frac{k}{m}}\ ,\quad b = 2mZ\omega_0\ ,\quad A_s = \frac{F_0}{k}$$

We also define the *frequency ratio*,

$$r = \frac{\omega}{\omega_0}$$

Then the magnification factor is

$$D_s = \frac{A}{A_s} = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + \left(2rZ\right)^2}} \tag{5}$$

and the phase angle is

$$\phi = \arctan\frac{2rZ}{1 - r^2} \tag{6}$$

What is the frequency, $\omega_{\rm max}$, say, that gives us the greatest amplitude? Or to put the question another way, what is the *resonant frequency*? We need to minimise the denominator in (5); we do this in the usual way by setting its first derivative with respect to r equal to zero and solving for r which will then give us $\omega_{\rm max}/\omega_0$.

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(1 - r^2\right)^2 + \left(2rZ\right)^2\right] = 0$$

leads to

$$\omega_{\max} = \omega_0\sqrt{1-2Z^2} \tag{7}$$

which is neither ω_0, nor $\omega_D = \omega_0\sqrt{1-Z^2}$.

What is the *maximum ampltitude*; $A_{\max}$, say? Put (7) into (5) and neglect Z^4 when compared to Z^2. We find

$$\frac{A_{\max}}{A_s} = \frac{1}{2Z} = \frac{m\omega_0}{b} \approx Q$$

which is the "quality factor", and using $A_s = F_0/k$ and $\omega_0^2 = k/m$ we get

$$A_{\max} = \frac{F_0}{b\omega_0}$$

On page 9 (below) are two graphs I've taken from `wikipedia` showing a set of resonance curves and phase angles for a driven damped oscillator. On the abscissa is plotted the frequency ratio, r. They use the phrase "amplification ratio" for the magnification factor and have used the symbol ζ for the damping factor, Z. The first is essentially a plot of equation (5). In the second, note how in the case of the undamped forced oscillator there is an abrupt change from in phase to 180° out of phase at r goes through one, as we discuss on page 5 of these notes. Note how the frequency $\omega_{\max}$ is always smaller than the natural frequency ω_0 but appears to approach it as the peak becomes narrower, that is, the damping becomes less.

There are three interesting cases.

(i) If $r \ll 1$ the driving frequency is much smaller than the natural frequency of the oscillator,

$$\omega \ll \omega_0$$

Then the dynamic amplitude is close to the static amplitude,

$$A \approx A_s$$

and the phase difference is

$$\phi \approx \arctan 0 = 0$$

so the displacement and force are in phase.

(ii) If $r \approx 1$ then

$$\omega \approx \omega_0$$

and

$$\frac{A}{A_s} \approx \frac{1}{2Z} \approx Q \text{ , the quality factor}$$

Also,

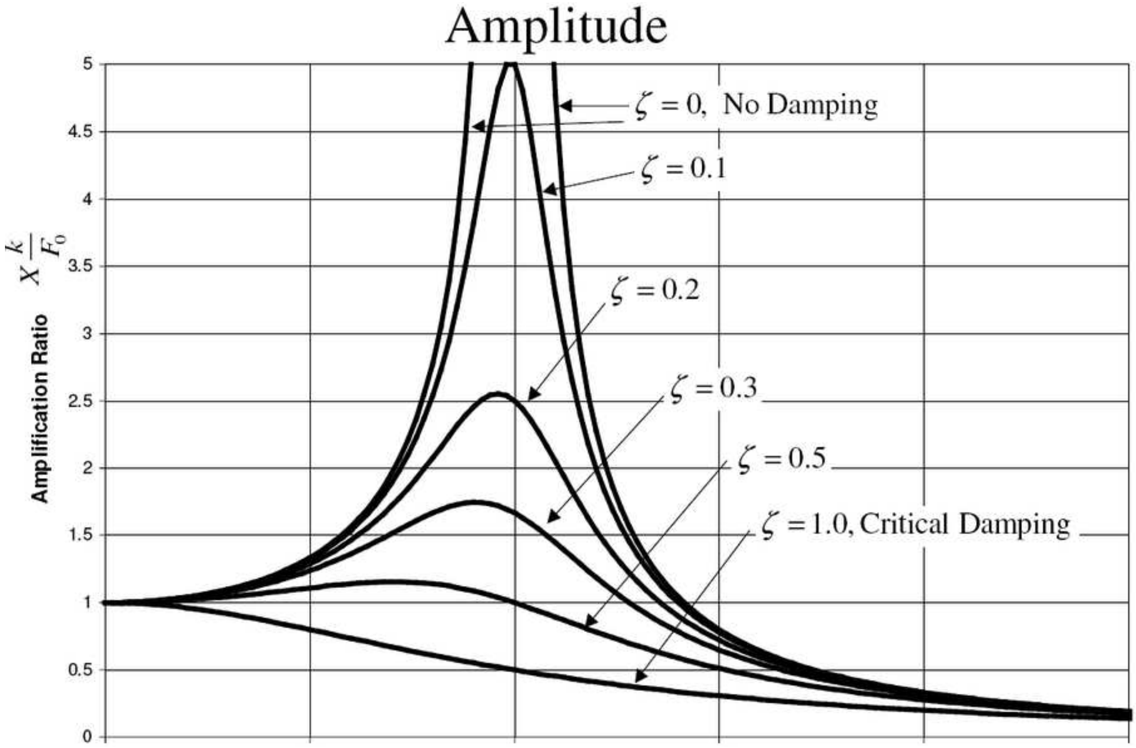
$$\phi \approx \arctan \infty = \frac{1}{2}\pi$$

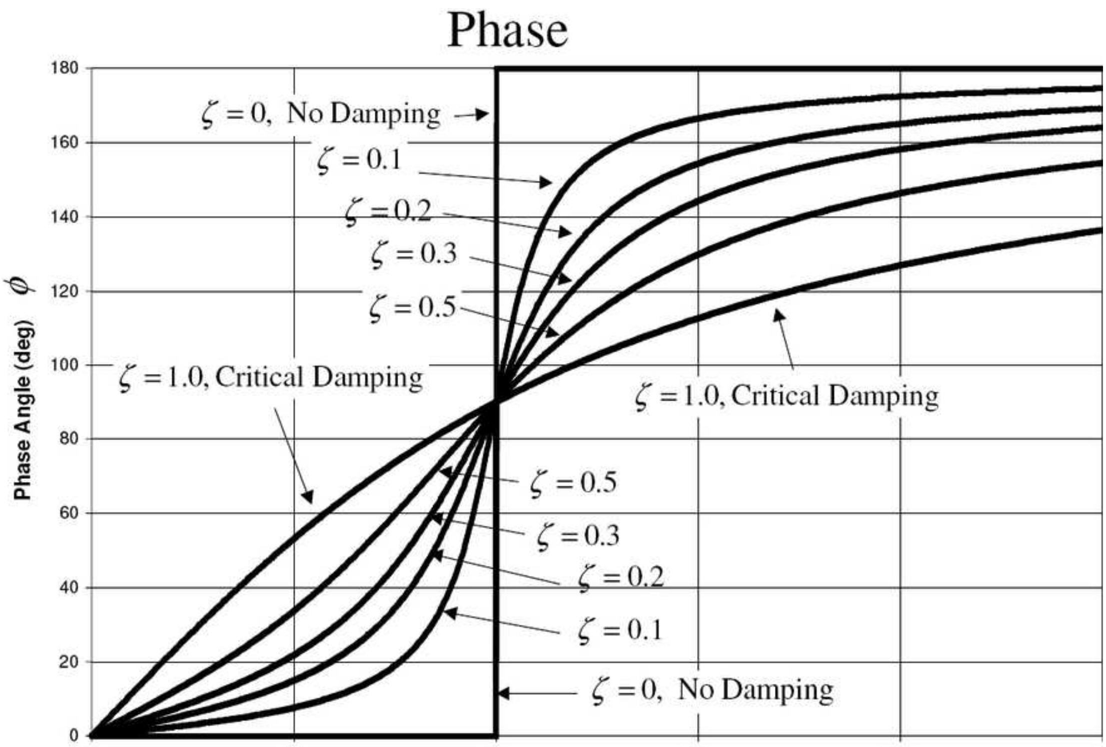
so the displacement and force are out of phase by 90°.

(iii) If $r \gg 1$, then $\omega \gg \omega_0$ and therefore

$$\frac{A}{A_s} \propto \frac{\omega_0^2}{\omega^2} = \frac{1}{r^2}$$

which is the shape of the high frequency tail of the resonance curve. The displacement and force are out of phase by 180°, for the same reason as given on page 6 for the driven undamped oscillator.


Amplitude
Amplification Ratio $X\frac{k}{F_0}$
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
$\zeta = 0$, No Damping
$\zeta = 0.1$
$\zeta = 0.2$
$\zeta = 0.3$
$\zeta = 0.5$
$\zeta = 1.0$, Critical Damping


Phase
Phase Angle (deg) ϕ
180
160
140
120
100
80
60
40
20
0
$\zeta = 0$, No Damping
$\zeta = 0.1$
$\zeta = 0.2$
$\zeta = 0.3$
$\zeta = 0.5$
$\zeta = 1.0$, Critical Damping
$\zeta = 1.0$, Critical Damping
$\zeta = 0.5$
$\zeta = 0.3$
$\zeta = 0.2$
$\zeta = 0.1$
$\zeta = 0$, No Damping

2. The LCR circuit

We will discuss an electrical circuit called LCR which is a resistor, an inductor and a capacitor connected in series to an AC power supply. This is a *resonant circuit.* First we study the three components separately.

2.1 The R-circuit

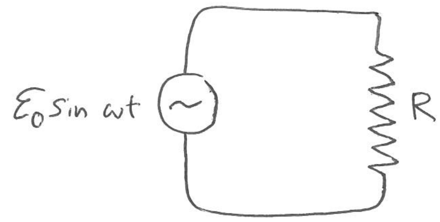


FIGURE 1

In the circuit in figure 1, $\mathcal{E}_0$ is the peak e.m.f. (voltage); ω is the driving angular frequency; t is time; R is the resistance in ohms. If I is the current, then

$$I = \frac{\mathcal{E}_0}{R} \sin \omega t \tag{2.1}$$

by Ohm's law. The current is *in phase with* the voltage

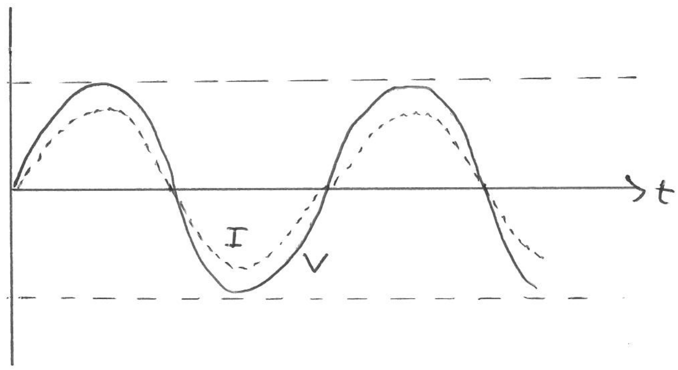


FIGURE 2

2.2 The C-circuit

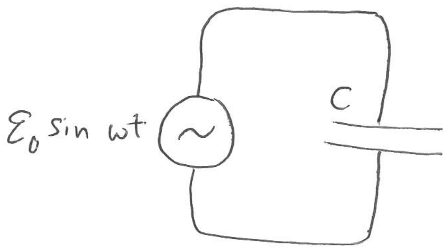


FIGURE 3

The capacitance is

$$C = \frac{Q}{V}$$

so the voltage is

$$\frac{Q}{C} = \mathcal{E}_0 \sin \omega t \tag{2.2}$$

Current is

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = C\mathcal{E}_0 \omega \cos \omega t$$

That is

$$I = \frac{\mathcal{E}_0}{1/\omega C} \sin(\omega t + \frac{1}{2}\pi) = \frac{\mathcal{E}_0}{X_C} \sin(\omega t + \frac{1}{2}\pi)$$

so the current *leads* the voltage by a phase angle of 90° as shown in figure 4.

$$X_C = \frac{1}{\omega C}$$

is called the *capacitive reactance.*

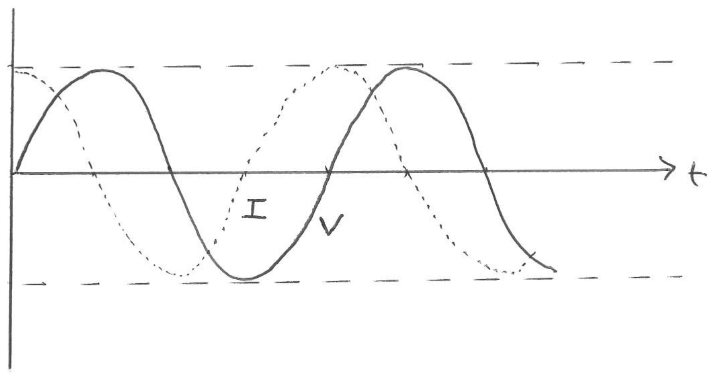


FIGURE 4

2.3 The L-circuit

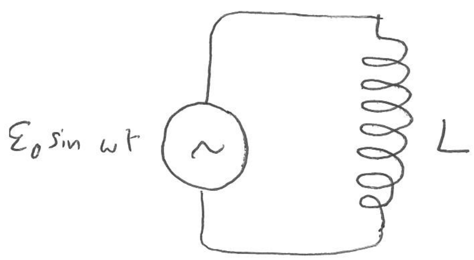


FIGURE 5

An inductor is a coil that produces a back e.m.f. as a magnetic field is grown inside the coil. The back e.m.f. is proportional to the rate of increase of current, the constant of proportionality is the *inductance* L. The back e.m.f. is $-L\dot{I}$ by Lenz's law and by Kirchhoff's loop law the back e.m.f. is $-\mathcal{E}$, then,

$$\mathcal{E}_0 \sin \omega t = L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{2.3}$$

that is,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathcal{E}_0}{L} \sin \omega t$$

and by integration,

$$\begin{aligned} I &= -\frac{\mathcal{E}_0}{\omega L}\cos\omega t \\ &= \frac{\mathcal{E}_0}{\omega L}\sin(\omega t - \frac{1}{2}\pi) \\ &= \frac{\mathcal{E}_0}{X_L}\sin(\omega t - \frac{1}{2}\pi) \end{aligned}$$

Now the voltage *leads* the current, or if you prefer the current *trails* the voltage.

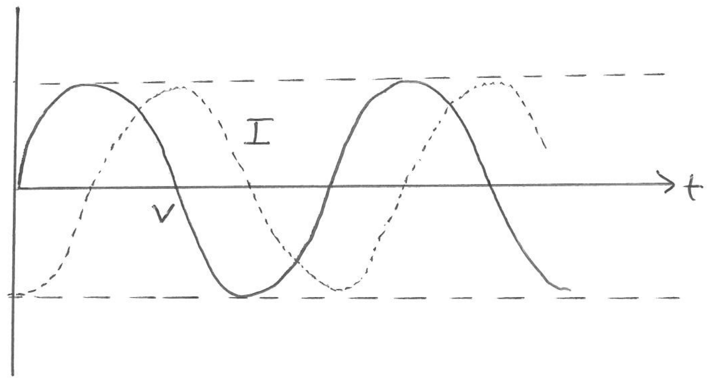


FIGURE 6

and

$$X_L = \omega L$$

is called the *inductive reactance.*

2.4 The LCR circuit

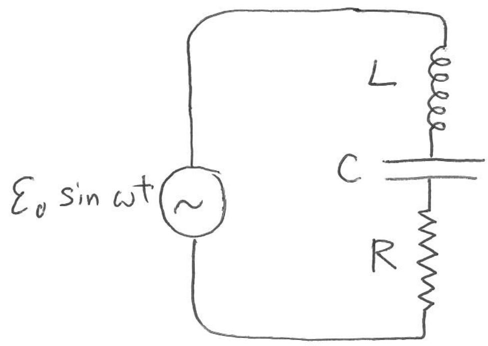


FIGURE 7

We now combine equations (2.1), (2.2) and (2.3)

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI + \frac{Q}{C} = \mathcal{E}_0 \sin\omega t$$

and since

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}$$

this is

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = \mathcal{E}_0 \sin\omega t$$

or

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{CL}Q = \frac{\mathcal{E}_0}{L}\sin\omega t$$

Compare this with the equation (4a), p. 5, which is the equation of motion of a driven, damped mechanical oscillator,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2Z_m\omega_0\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{F_0}{m}\sin\omega t$$

In the case of the LCR circuit we will write

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + 2\mathcal{Z}\omega_0\frac{\mathrm{d}Q}{\mathrm{d}t} + \omega_0^2 Q = \frac{\mathcal{E}_0}{L}\sin\omega t$$

and we use the symbol Z_m for the damping ratio of the mechanical device and $\mathcal{Z}$ for the damping ratio of the LCR circuit.

The following table shows the correspondence between the parameters of the two devices.

	damped mass and spring	LCR circuit
inertial element	m	L
stiffness	k	$1/C$
damping coefficient	b	R
damping ratio	$\frac{1}{2}b/\sqrt{mk}$	$\frac{1}{2}R\sqrt{C/L}$
static amplitude A_s	F_0/k	$\mathcal{E}_0 C$
quality factor $\mathcal{Q}$	$\sqrt{mk}/b$	$\frac{1}{R}\sqrt{L/C}$
natural frequency ω_0	$\sqrt{k/m}$	$1/\sqrt{LC}$
$A_{\max} = A_s\mathcal{Q}$	$F_0/b\omega_0$	$\mathcal{E}_0/R\omega_0$

We can use this table as a "dictionary" to translate the solution subsection 1.3 into the physics of the present situation. There, we had

$$x = A\sin(\omega t - \phi)$$

with

$$\frac{A}{A_s} = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2Z_m\frac{\omega}{\omega_0}\right)^2}}$$

and

$$A_s = \frac{F_0}{k}$$

For convenience in what follows, we will shift the phase and write for the time dependence of the charge, Q,

$$Q = -A\cos(\omega t - \phi)$$

This is of course just as good a solution of the differential equation for Q. The shift of phase in our choice of solution of the differential equation does not affect the amplitude, so we still have A/A_s as above, but after substituting $\mathcal{Z}$ for Z_m and using $A_s = \mathcal{E}_0 C$. To obtain the current, we differentiate the charge with respect to time,

$$\begin{aligned} I = \frac{\mathrm{d}Q}{\mathrm{d}t} &= A\omega\sin(\omega t - \phi) \\ &= I_0 \sin(\omega t - \phi) \end{aligned} \tag{2.4}$$

with†

$$I_0 = \frac{\omega\mathcal{E}_0 C}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\mathcal{Z}\frac{\omega}{\omega_0}\right)^2}}$$

and after a lot of easy algebra, and using $\mathcal{Z} = \frac{1}{2}R\sqrt{C/L}$ and $\omega_0 = 1/\sqrt{LC}$, this turns into

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}} \tag{2.5}$$

The phase angle turns out to be‡

$$\begin{aligned} \phi &= \arctan\frac{\omega L - \frac{1}{\omega C}}{R} \\ &= \arctan\frac{\text{reactance}}{\text{resistance}} \end{aligned} \tag{2.6}$$

in which by "reactance" I mean the the inductive reactance take away the capacitive reactance. Now you are going to see why I have used new symbols Z_m and $\mathcal{Z}$ for the damping ratios (as well as $\mathcal{Q}$ for quality factor, so as not to confuse it with Q for charge)—everybody writes equation (2.5) as

$$I_0 = \frac{\mathcal{E}_0}{Z}$$

That is, the peak current equals the peak voltage divided by, not *resistance* as in a direct current circuit but, *impedance*. Impedance is always given the symbol Z. The impedance of the LCR circuit, according to equation (2.5) is given by

$$Z^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 + R^2$$

---

† I chose the $-\cos$ solution so that we now have $\mathcal{E} = \mathcal{E}_0 \sin\omega t$ and $I = I_0\sin(\omega t - \phi)$ and this instantly identifies ϕ as the phase difference between the current and the voltage, which we what we want to know.

‡ Compare with page 6 of these notes: after changing $x = A\sin(\omega t - \phi)$ into $x = -A\cos(\omega t - \phi)$. In that case $\tan\phi = (r^2 - 1)/2rZ_m$ with $r = \omega/\omega_0$; then use the dictionary.

and we remember that ωL is the inductive reactance and $1/\omega C$ is the capacitive reactance. So

$$\begin{aligned} Z^2 &= (X_L - X_C)^2 + R^2 \\ &= (\text{reactance})^2 + (\text{resistance})^2 \end{aligned}$$

Depending on the relative sizes of the inductive and capacitive reactances, the total reactance may be positive or negative. We must therefore interpret Z^2 as the "modulus squared" of a complex number:†

$$Z^* Z = |Z|^2$$

and

$$Z = R + i\,(X_L - X_C)$$

is called the *complex impedance* of the LCR circuit. We can also write this as

$$Z = |Z|\, e^{i\phi}$$

and plot Z in the Argand diagram

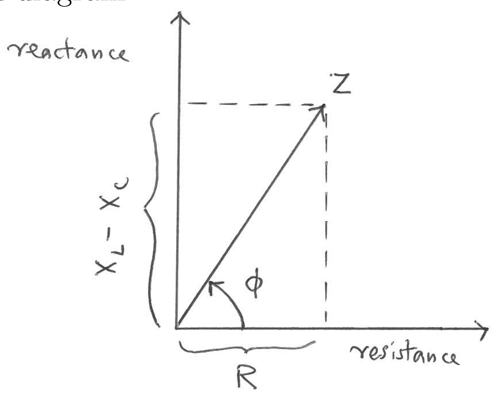


FIGURE 8

We then see that the phase is exactly as in equation (2.6), namely,

$$\tan\phi = \frac{\text{reactance}}{\text{resistance}}$$

This is very important because we now see that however difficult the maths has been in all this development, for whatever LCR circuit we construct once we know the resistance, capacitance and inductance of the three elements and the driving angular frequency, ω, of the a.c. power supply, then we can easily calculate the reactance and then with a diagram like figure 8, we have a simple *graphical construction* to find the impedance and the phase difference—that is, whether and by how much the current leads or trails the voltage in our LCR circuit.

† We need to do this because we wish to write $Z = a + b$, say, that is the sum of a reactive part and a resistive part; while what we have is $Z^2 = A^2 + B^2$, say. If Z is real, then we have $Z^2 = (a+b)^2 = a^2 + b^2 + 2ab$ which is not in the form $a^2 + b^2$ so this doesn't work. But if Z is *complex* we can write $Z = a + ib$ and then $|Z|^2 = (a - ib)(a + ib) = a^2 + b^2$ as we require. So we need to insist that either the reactance or the resistance is "imaginary".

We now write down the oscillating e.m.f. as in figure 7,

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_0 \sin\omega t \\ &= \operatorname{Im}\mathcal{E}_0\, e^{i\omega t}\end{aligned} \tag{2.7}$$

in which Im means "imaginary part of", and so the oscillating current in the LCR circuit is

$$\begin{aligned}I = \frac{\mathcal{E}}{Z} &= \operatorname{Im}\frac{\mathcal{E}_0}{|Z|}e^{-i\phi}\, e^{i\omega t} \\ &= \frac{\mathcal{E}_0}{|Z|}\sin(\omega t - \phi) \\ &= I_0 \sin(\omega t - \phi)\end{aligned} \tag{2.4}$$

which is the same as equation (2.4). This is why we wanted a "sine" solution for the current and hence needed a "minus cosine" solution for the charge at the top of page 5. By comparison with equation (2.7), the angle ϕ determines the angle by which the current trails (or leads if $\phi < 0$) the voltage. We can think of this as arising in a diagram like figure 9 which provides a graphical means to find the phase relation between voltage and current in an LCR circuit, given the values of L, R and C.

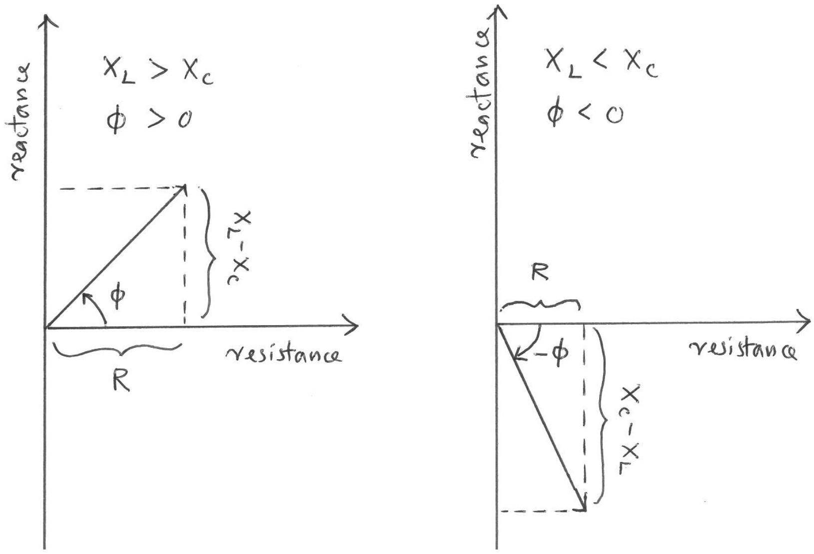


FIGURE 9

This illustrates how the phase is zero in a purely resistive circuit as in figure 2. The phase is positive or negative depending whether the reactance is positive or negative, or in other words, whether the inductive reactance is greater or less than the capacitive reactance. In a purely *capacitive* circuit we see that $\phi = -\frac{1}{2}\pi = -90°$; and in a purely *inductive* circuit we see that $\phi = +\frac{1}{2}\pi = +90°$ which is entirely consistent with figures 4 and 6.

The LCR circuit is a *resonant oscillator* just as is its mechanical counterpart. The angular frequency ω_0 is the natural frequency of the *undamped* circuit, that is a circuit with $R = 0$, so ω_0 is the solution of the equation[†]

$$\omega L - \frac{1}{\omega C} = 0$$

that is,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The quality factor is

$$\mathcal{Q} = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

and the static amplitude is $A_s = \mathcal{E}_0 C$ which is the charge stored in the capacitor if the frequency is zero so that the circuit becomes a DC circuit. A circuit with a high $\mathcal{Q}$ has a narrower resonance peak or "bandwidth". This is how a radio is tuned. The capacitance of a resonant circuit is varied until the resonant frequency matches the frequency of the signal being sought. By exploiting a narrow resonance peak, signals at nearby frequencies do not affect the current in the circuit.

Figure 10 shows a typical resonance and power output curve for an LCR circuit. You should be able to identify A_s and $A_{\max}$ in the left hand graph. In the right hand graph, $\Delta\omega$ is the bandwidth. There's a good `wikipedia` page on the LCR circuit (`en.wikipedia.org/wiki/RLC_circuit`). Note that they call it an RLC circuit. Also in your textbooks, so when you look it up in the index, try under "R" and "C" as well as "L".

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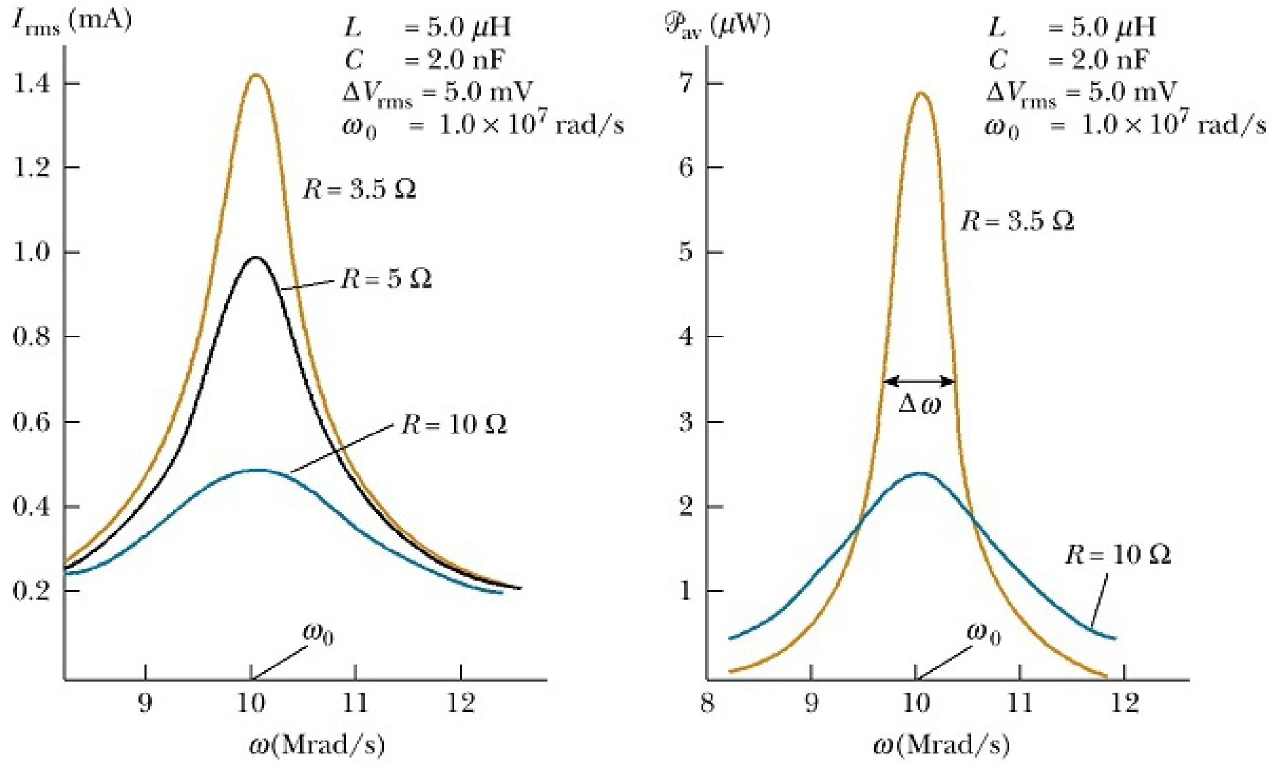
† The reason for this is that I am looking for the value of ω that *maximises* the peak current in equation (2.5) in the absence of damping, that is, $R = 0$. So I need to *minimise* the denominator. Actually it is smallest when it is zero. And this means that at resonance the undamped oscillator has infinite amplitude. This is also the case for the mass on the spring as we see in the subsection 1 of these notes. Of course in real life there is no such thing as a totally undamped oscillator—there is always some damping—but at resonance the amplitude can be very large in a high $\mathcal{Q}$ LCR circuit (or any other oscillator with a large quality factor).



I_{rms} (mA)
L = 5.0 μH
C = 2.0 nF
ΔV_{rms} = 5.0 mV
ω_0 = 1.0×10^7 rad/s
$R = 3.5\ \Omega$
$R = 5\ \Omega$
$R = 10\ \Omega$
ω_0
ω(Mrad/s)
$\mathscr{P}_{av}$ (μW)
L = 5.0 μH
C = 2.0 nF
ΔV_{rms} = 5.0 mV
ω_0 = 1.0×10^7 rad/s
$R = 3.5\ \Omega$
$\Delta\omega$
$R = 10\ \Omega$
ω_0
ω(Mrad/s)

FIGURE 10